Final Exam (Answers)

Exercise 1 [4pt]
Let (X, Y) have density

$$f(x,y) = \begin{cases} xe^{-x(y+1)} & \text{if } x \geq 0 \text{ and } y \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

1. Check that f is a density on $\mathbb{R}^2$.
2. Find the marginal density of X.
3. Prove that $\mathbb{E}[Y|X] = 1/X$.

Answers.

1. *f is obviously non-negative, let us compute*

$$\begin{aligned} \int\int f(x,y)dxdy &= \int_{x=0}^{+\infty} \left(\int_{y=0}^{\infty} xe^{-x(y+1)} dy \right) dx \\ &= \int_{x=0}^{+\infty} x \left| -\frac{e^{-x(y+1)}}{x} \right|_{y=0}^{\infty} dx \\ &= \int_{x=0}^{+\infty} x \frac{e^{-x}}{x} dx = \int_{x=0}^{+\infty} e^{-x} dx = 1. \end{aligned}$$

2. *We integrate w.r.t. y. For $x \geq 0$,*

$$f_X(x) = \int_{y=0}^{\infty} xe^{-x(y+1)} dy = x \left| -\frac{e^{-x(y+1)}}{x} \right|_{y=0}^{\infty} = x\frac{e^{-x}}{x} = e^{-x}.$$

X follows the exponential distribution with parameter 1.

3. *Assume that $X > 0$,*

$$\begin{aligned} \mathbb{E}[Y|X] &= \frac{\int_{y=0}^{\infty} yf(X,y)dy}{f_X(X)} \\ &= \frac{\int_{y=0}^{\infty} yXe^{-X(y+1)}dy}{e^{-X}} \\ &= \frac{X}{e^{-X}} \int_{y=0}^{\infty} ye^{-X(y+1)}dy. \end{aligned}$$

Let us integrate by parts the latter integral by setting $u(y) = y$, $v'(y) = e^{-X(y+1)}$, so that

$$\begin{array}{rclcl} \int_{y=0}^{\infty} ye^{-X(y+1)}dy &=& \left|-ye^{-X(y+1)}/X\right|_{y=0}^{\infty} &-& \int_{y=0}^{\infty}(-e^{-X(y+1)}/X)dy \\ &=& 0 &-& \frac{1}{X}\left|-e^{-X(y+1)}/X\right|_{y=0}^{\infty} \\ &=& 0 &+& \frac{1}{X^2}e^{-X}. \end{array}$$

This shows that $\mathbb{E}[Y|X] = 1/X$.

Exercise 2 [5pts]
We say that X follows the Cauchy distribution if the characteristic function of X is

$$\Phi_X(t) = \mathbb{E}[e^{itX}] = \exp(-|t|).$$

Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables following the Cauchy distribution.

1. Find the law of $\dfrac{X_1 + X_2}{2}$.

2. Set $S_n = \sum_{k=1}^n X_k$. For which $\alpha > 0$ does the sequence $\left(\dfrac{S_n}{n^\alpha}\right)$ converge in law to 0?

We admit that X has density

$$f(x) = \frac{1}{\pi(1+x^2)} \text{ for all } x \in \mathbb{R}$$

(you don't need to check that f is a density).

3.∗ Prove that if X has the Cauchy distribution, then $Y = \frac{1}{X}$ has the Cauchy distribution also.

Answers.

1. *Let t be a real number,*

$$\begin{aligned} \mathbb{E}[\exp(it\,(X_1+X_2)\,/2)] &= \mathbb{E}[\exp(itX_1/2)]\mathbb{E}[\exp(itX_2/2)] \qquad \textit{(by independence)} \\ &= \exp(-|t/2|)\exp(-|t/2|) \\ &= \exp(-|t|/2 - |t|/2) = \exp(-|t|), \end{aligned}$$

i.e. $(X_1+X_2)/2$ *has the characteristic function of the Cauchy distribution.*

2. *Let t be real,*

$$\begin{aligned} \mathbb{E}\left[\exp(itS_n/n^\alpha)\right] &= \mathbb{E}\left[\exp\left(\frac{it}{n^\alpha}(X_1+\cdots+X_n)\right)\right] \\ &= \mathbb{E}\left[\exp\left(\frac{it}{n^\alpha}X_1\right)\right]^n \\ &= \exp(-|t/n^\alpha|)^n = \exp\left(-|t|n^{1-\alpha}\right). \end{aligned}$$

Now, saying that S_n/n^α converges in law to 0 amounts to say that

$$\exp\left(-|t|n^{1-\alpha}\right) \overset{n\to+\infty}{\to} \mathbb{E}[\exp(it\times 0)] = 1$$

for all t. This holds when $n^{1-\alpha} \to 0$, i.e. *$\alpha > 1$. Finally,*

$$\left(\frac{S_n}{n^\alpha}\right) \overset{\textit{(law)}}{\to} 0 \qquad \textit{iff } \alpha > 1.$$

3. *Let ϕ be a bounded and continuous function.*

$$\begin{aligned} \mathbb{E}[\phi(Y)] = \mathbb{E}[\phi(1/X)] &= \int_{-\infty}^{+\infty} \frac{\phi(1/x)}{\pi(1+x^2)}dx \\ &= \int_{-\infty}^{0} \frac{\phi(1/x)}{\pi(1+x^2)}dx + \int_{0}^{+\infty} \frac{\phi(1/x)}{\pi(1+x^2)}dx \end{aligned}$$

We make change of variables[1]

$$y = 1/x, \qquad x = 1/y, \qquad dx = -dy/y^2.$$

We obtain

$$\begin{aligned}\mathbb{E}[\phi(Y)] &= \int_{-\infty}^{0} \frac{\phi(y)}{\pi(1+1/y^2)}\frac{dy}{y^2} + \int_{0}^{+\infty} \frac{\phi(y)}{\pi(1+1/y^2)}\frac{dy}{y^2} \\ &= \int_{-\infty}^{+\infty} \frac{\phi(y)}{\pi(y^2+1)} dy \\ &= \mathbb{E}[\phi(X)],\end{aligned}$$

and thus X, Y have the same distribution.

Exercise 3 [4pts]
Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. random variables following the normal distribution $\mathcal{N}(0,1)$. We set $S_n = \sum_{k=1}^{n} X_k$ and, for any constants a, b,

$$Z_n = \exp(aS_n - bn).$$

1. Prove that, for all real t, $\mathbb{E}[\exp(tX_1)] = \exp(t^2/2)$.
   (Hint: you can use the fact that $tx - \frac{x^2}{2} = -\frac{(x-t)^2}{2} + \frac{t^2}{2}$.)
2. Use question 1. to prove that

$$\left((Z_n)_{n\geq 0} \overset{L^p}{\to} 0\right) \qquad \text{if and only if} \qquad p < 2b/a^2.$$

3.∗ Prove that if $b > 0$, $(Z_n)_{n\geq 0} \overset{\text{a.s.}}{\to} 0$.
(Hint: no computation is needed here.)

Answers.

1.

$$\begin{aligned}\mathbb{E}[\exp(tX_1)] &= \int_{-\infty}^{+\infty} e^t x \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx \\ &= \int_{-\infty}^{+\infty} \frac{\exp\left(-\frac{(x-t)^2}{2} + \frac{t^2}{2}\right)}{\sqrt{2\pi}} dx \\ &= \exp(t^2/2) \int_{-\infty}^{+\infty} \frac{\exp(-(x-t)^2/2)}{\sqrt{2\pi}} dx\end{aligned}$$

but the latter integral is 1 since this is the density of a $\mathcal{N}(t,1)$.

2.

$$\begin{aligned}\mathbb{E}\left[(Z_n)^p\right] &= \mathbb{E}[\exp(apS_n - bpn)] \\ &= \mathbb{E}[e^{apX_1}]^n \exp(-bpn) \\ &= \exp(n(ap)^2/2)\exp(-bpn) = \exp\left(n(a^2p^2/2 - bp)\right).\end{aligned}$$

This goes to 0 iff $a^2p^2/2 - bp < 0$, i.e. *$p < 2b/a^2$.*

[1] This is why we cut the integral into two pieces: $y = 1/x$ is not monotonous on $\mathbb{R}$.

3.

$$Z_n = \exp\Big(n \underbrace{(aS_n/n - b)}_{\to -b \text{ a.s.}}\Big)$$

(by the Law of Large Numbers). Z_n goes to zero if $b > 0$.

Exercise 4 [4pts]
Let X have cumulative distribution function

$$F_X(t) = \mathbb{P}(X \le t) = \begin{cases} 0 & \text{if } t < 0, \\ 4t^3 - 3t^4 & \text{if } 0 \le t \le 1, \\ 1 & \text{if } t > 1. \end{cases}$$

1. Compute the density of X.

Let $X_1, X_2, \dots$ be i.i.d. random variables with the same law as X. Let

$$M_n = \min\{X_1, \dots, X_n\}$$

be the smallest value among the n first X_k's.

2. Find the cumulative distribution function of M_n.
3. Let Z have cumulative distribution function

$$F_Z(t) = 1 - \exp(-4t^3) \qquad \text{for } t \ge 0$$

(and 0 otherwise). Prove that $(n^{1/3}M_n) \overset{\text{(law)}}{\to} Z$.

Answers.

1. *Let $0 \le t \le 1$, and let f be the density of X,*

$$f(t) = F_X'(t) = 12t^2 - 12t^3 = 12t^2(1-t).$$

2. *Let $0 \le t \le 1$,*

$$\begin{aligned} \mathbb{P}(M_n \ge t) &= \mathbb{P}(X_1 \ge t \cap X_2 \ge t \cap \dots \cap X_n \ge t) \\ &= \mathbb{P}(X_1 \ge t)^n = (1 - 4t^3 - 3t^4)^n, \end{aligned}$$

so that

$$F_{M_n}(t) = 1 - (1 - 4t^3 - 3t^4)^n$$

if $0 \le t \le 1$, and 0 otherwise.

3. *We use cumulative distribution functions.*

$$\begin{aligned} \mathbb{P}(n^{1/3}M_n \ge t) &= 1 - \left(1 - 4\left(\tfrac{t}{n^{1/3}}\right)^3 - 3\left(\tfrac{t}{n^{1/3}}\right)^4\right)^n, \\ &= 1 - \left(1 - 4\tfrac{t^3}{n} + \mathrm{o}(1/n)\right)^n, \\ &\overset{n \to +\infty}{\to} 1 - \exp(-4t^3) \end{aligned}$$

(if $t \ge 0$). Since $t \mapsto 1 - \exp(-4t^3)$ is continuous, then this proves the convergence in distribution towards Z.

Exercise 5 [3pts]
Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables in L^4 (so that $X_i^2 \in L^2$). We assume that $\mathbb{E}[X_i] = 0$ and $\mathbb{E}[X_i^2] = 1$.
Set as usual $S_n = X_1 + X_2 + \cdots + X_n$.

1. Compute $\mathbb{E}\left[\,(S_{n+1})^2\;|S_n\right]$.

2.* Let $n < m$ be integers, prove that $\mathrm{Cov}(S_n, S_m) = n$.

Answers.

1.

$$\begin{array}{rcccccc}
\mathbb{E}\left[\,(S_{n+1})^2\;|S_n\right] & = & \mathbb{E}\left[(S_n + X_{n+1})^2|S_n\right] & & & & \\
& = & \mathbb{E}\left[S_n^2\;|S_n\right] & + & \mathbb{E}\left[X_{n+1}^2\;|S_n\right] & + & \mathbb{E}\left[2X_{n+1}S_n\;|S_n\right] \\
& = & S_n^2 & + & \mathbb{E}\left[X_{n+1}^2\right] & + & 2S_n\mathbb{E}\left[X_{n+1}\right] \\
& = & S_n^2 & + & \mathbb{E}\left[X_{n+1}^2\right] & + & 0 \\
& = & S_n^2 & + & 1. & &
\end{array}$$

2.

$$\begin{aligned}
\mathbb{E}[S_n S_m] &= \mathbb{E}[S_n(S_m - S_n + S_n)] \\
&= \mathbb{E}[S_n(S_m - S_n)] + \mathbb{E}[S_n^2] \\
&= \mathbb{E}[S_n]\mathbb{E}[(S_m - S_n)] + \mathbb{E}[S_n^2] = 0 + n.
\end{aligned}$$

since $S_m - S_n = X_{n+1} + X_{n+2} + \cdots + X_m$ is independent from S_n. Now,

$$\mathrm{Cov}(S_n, S_m) = \mathbb{E}[S_n S_m] - \mathbb{E}[S_n]\mathbb{E}[S_m] = n.$$

Exercise 6 [optional: 2pts]
Let Z be a discrete random variable with values in $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$. Prove that

$$Z \sim \mathrm{Poisson}(\lambda) \qquad \Leftrightarrow \qquad \lambda\mathbb{E}[h(Z+1)] = \mathbb{E}[Zh(Z)] \text{ for all bounded } h : \mathbb{Z}_+ \to \mathbb{R}.$$

(Hint: For the $\Leftarrow$ part, find a recursion formula for the sequence $p_k = \mathbb{P}(Z = k)$.)

Answers.

We first prove that the "only if part". Assume that Z follows the Poisson distribution with parameter λ, and take a bounded $\phi : \mathbb{Z} \to \mathbb{R}$.

$$\begin{aligned}
\mathbb{E}[\phi(Z+1)] &= \sum_{k\geq 0} \phi(k+1) e^{-\lambda} \frac{\lambda^k}{k!} \\
&= \sum_{j\geq 1} \phi(j) e^{-\lambda} \frac{\lambda^{j-1}}{(j-1)!} \text{ (by change of variable } j \leftarrow k+1) \\
&= 1/\lambda \sum_{j\geq 1} \phi(j) e^{-\lambda} j \frac{\lambda^j}{j!} \\
&= 1/\lambda \sum_{j\geq 0} \phi(j) e^{-\lambda} j \frac{\lambda^j}{j!} \text{ (term } j = 0 \text{ cancels)},
\end{aligned}$$

and the "only if part" is proved.
Assume now that $\lambda\mathbb{E}[\phi(Z+1)] = \mathbb{E}[Z\phi(Z)]$ for all ϕ. Apply this with

$$\phi(z) = \mathbf{1}_{z=k},$$

we obtain

$$\begin{aligned} \lambda\mathbb{E}[\mathbf{1}_{Z+1=k}] &= \mathbb{E}[Z\mathbf{1}_{Z=k}] \\ \lambda\mathbb{P}(Z+1=k) &= \mathbb{E}[k\mathbf{1}_{Z=k}] \\ \lambda\mathbb{P}(Z+1=k) &= k\mathbb{P}(Z=k). \end{aligned}$$

This is true for all $k \geq 1$, and then

$$p_k = \frac{\lambda}{k} p_{k-1} = \frac{\lambda}{k} \times \left(\frac{\lambda}{k-1} p_{k-2} \right) = \cdots = \frac{\lambda^k}{k!} p_0.$$

It remains to find p_0. We let you check that the only choice that makes $\sum_k p_k = 1$ is $p_0 = e^{-\lambda}$.