Final Exam (Answers)

Exercise 1 [4pt]
Let \((X,Y)\) have density

\[
f(x,y) = \begin{cases} 
xe^{-x(y+1)} & \text{if } x \geq 0 \text{ and } y \geq 0, \\
0 & \text{otherwise.}
\end{cases}
\]

1. Check that \(f\) is a density on \(\mathbb{R}^2\).
2. Find the marginal density of \(X\).
3. Prove that \(\mathbb{E}[Y|X] = 1/X\).

Answers.

1. \(f\) is obviously non-negative, let us compute

\[
\int \int f(x,y) \, dx \, dy = \int_{x=0}^{\infty} \left( \int_{y=0}^{\infty} xe^{-x(y+1)} \, dy \right) \, dx
\]

\[
= \int_{x=0}^{\infty} x \left. \frac{e^{-x(y+1)}}{-x} \right|_{y=0}^{\infty} \, dx
\]

\[
= \int_{x=0}^{\infty} x e^{-x} \, dx = \int_{x=0}^{\infty} e^{-x} \, dx = 1.
\]

2. We integrate w.r.t. \(y\). For \(x \geq 0\),

\[
f_X(x) = \int_{y=0}^{\infty} xe^{-x(y+1)} \, dy = x \left. \frac{e^{-x(y+1)}}{-x} \right|_{y=0}^{\infty} = \frac{e^{-x}}{x} = e^{-x}.
\]

\(X\) follows the exponential distribution with parameter 1.

3. Assume that \(X > 0\),

\[
\mathbb{E}[Y|X] = \frac{\int_{y=0}^{\infty} yf(X,y) \, dy}{f_X(X)} = \frac{\int_{y=0}^{\infty} X e^{-X(y+1)} \, dy}{e^{-X}}
\]

\[
= X \int_{y=0}^{\infty} ye^{-X(y+1)} \, dy.
\]

Let us integrate by parts the latter integral by setting \(u(y) = y\), \(v'(y) = e^{-X(y+1)}\), so that

\[
\int_{y=0}^{\infty} ye^{-X(y+1)} \, dy = \left. ye^{-X(y+1)} \right|_{y=0}^{\infty} - \int_{y=0}^{\infty} \frac{1}{X}e^{-X(y+1)} \, dy
\]

\[
= 0 - \frac{1}{X} \int_{y=0}^{\infty} e^{-X(y+1)} \, dy + \frac{1}{X^2} e^{-X}.
\]

This shows that \(\mathbb{E}[Y|X] = 1/X\).
Exercise 2 [5pts]
We say that $X$ follows the Cauchy distribution if the characteristic function of $X$ is
\[
\Phi_X(t) = \mathbb{E}[e^{itX}] = \exp(-|t|). \]

Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables following the Cauchy distribution.

1. Find the law of $\frac{X_1 + X_2}{2}$.

2. Set $S_n = \sum_{k=1}^{n} X_k$. For which $\alpha > 0$ does the sequence $\left( \frac{S_n}{n^\alpha} \right)$ converge in law to 0?

We admit that $X$ has density
\[
f(x) = \frac{1}{\pi(1 + x^2)} \quad \text{for all } x \in \mathbb{R} \]
(you don’t need to check that $f$ is a density).

3. Prove that if $X$ has the Cauchy distribution, then $Y = \frac{1}{X}$ has the Cauchy distribution also.

Answers.

1. Let $t$ be a real number,
\[
\mathbb{E}[\exp(it(X_1 + X_2)/2)] = \mathbb{E}[\exp(itX_1/2)]\mathbb{E}[\exp(itX_2/2)] \quad \text{(by independence)}
\]
\[
= \exp(-|t/2|) \exp(-|t/2|)
\]
\[
= \exp(-|t|/2 - |t|/2) = \exp(-|t|),
\]
i.e. $(X_1 + X_2)/2$ has the characteristic function of the Cauchy distribution.

2. Let $t$ be real,
\[
\mathbb{E}[\exp(itS_n/n^\alpha)] = \mathbb{E}\left[\exp\left(\frac{it}{n^\alpha}(X_1 + \cdots + X_n)\right)\right]
\]
\[
= \mathbb{E}\left[\exp\left(\frac{it}{n^\alpha}X_1\right)^n\right]
\]
\[
= \exp(-|t|/n^\alpha)^n = \exp(-|t|n^{1-\alpha}).
\]

Now, saying that $S_n/n^\alpha$ converges in law to 0 amounts to say that
\[
\exp(-|t|n^{1-\alpha}) \xrightarrow{n \to +\infty} \mathbb{E}[\exp(it \times 0)] = 1
\]
for all $t$. This holds when $n^{1-\alpha} \to 0$, i.e. $\alpha > 1$. Finally,
\[
\left( \frac{S_n}{n^\alpha} \right) \xrightarrow{(law)} 0 \quad \text{iff } \alpha > 1.
\]

3. Let $\phi$ be a bounded and continuous function.
\[
\mathbb{E}[\phi(Y)] = \mathbb{E}[\phi(1/X)] = \int_{-\infty}^{+\infty} \frac{\phi(1/x)}{\pi(1 + x^2)} \, dx
\]
\[
= \int_{-\infty}^{0} \frac{\phi(1/x)}{\pi(1 + x^2)} \, dx + \int_{0}^{+\infty} \frac{\phi(1/x)}{\pi(1 + x^2)} \, dx
\]
We make change of variables\(^1\)
\[ y = \frac{1}{x}, \quad x = \frac{1}{y}, \quad dx = -\frac{dy}{y^2}. \]

We obtain
\[
\mathbb{E}[\phi(Y)] = \int_{-\infty}^{0} \frac{\phi(y)}{\pi(1 + 1/y^2)} \frac{dy}{y^2} + \int_{0}^{+\infty} \frac{\phi(y)}{\pi(1 + 1/y^2)} \frac{dy}{y^2} - \int_{-\infty}^{+\infty} \frac{\phi(y)}{\pi(y^2 + 1)} dy = \mathbb{E}[\phi(X)],
\]
and thus \(X, Y\) have the same distribution.

**Exercise 3** [4pts]
Let \((X_n)_{n \geq 1}\) be a sequence of i.i.d. random variables following the normal distribution \(N(0, 1)\). We set \(S_n = \sum_{k=1}^{n} X_k\) and, for any constants \(a, b\),
\[ Z_n = \exp(aS_n - bn). \]

1. Prove that, for all real \(t\), \(\mathbb{E}[\exp(tX_1)] = \exp(t^2/2)\).
   (Hint: you can use the fact that \(tx - \frac{x^2}{2} = -(\frac{x-t}{2})^2 + \frac{t^2}{2}\).)

2. Use question 1. to prove that
   \[
   \left( (Z_n)_{n \geq 0} \xrightarrow{L^p} 0 \right) \quad \text{if and only if} \quad p < 2b/a^2.
   \]

3. Prove that if \(b > 0\), \((Z_n)_{n \geq 0} \xrightarrow{a.s.} 0.\)
   (Hint: no computation is needed here.)

**Answers.**

1. \[
\mathbb{E}[\exp(tX_1)] = \int_{-\infty}^{+\infty} e^t x \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx = \int_{-\infty}^{+\infty} \exp\left(-\frac{(x-t)^2 + t^2}{2}\right) dx
\]
   \[
   = \exp(t^2/2) \int_{-\infty}^{+\infty} \frac{\exp(- (x - t)^2/2)}{\sqrt{2\pi}} dx
   \]
   but the latter integral is 1 since this is the density of a \(N(t, 1)\).

2. \[
\mathbb{E}[(Z_n)^p] = \mathbb{E}[\exp(apS_n - bpn)] = \mathbb{E}[e^{apX_1}]^n \exp(-bpn)
\]
   \[
   = \exp(n(a^2 p^2/2 - bp)) = \exp(n(a^2 p^2/2 - bp)).
   \]
   This goes to 0 iff \(a^2 p^2/2 - bp < 0\), i.e. \(p < 2b/a^2\).

\(^1\)This is why we cut the integral into two pieces: \(y = 1/x\) is not monotonous on \(\mathbb{R}\).
\[ Z_n = \exp \left( n \frac{aS_n}{n-b} \right) \rightarrow b \text{ a.s.} \]

(by the Law of Large Numbers). \( Z_n \) goes to zero if \( b > 0 \).

**Exercise 4** [4pts]

Let \( X \) have cumulative distribution function
\[
F_X(t) = \mathbb{P}(X \leq t) = \begin{cases} 
0 & \text{if } t < 0, \\
4t^3 - 3t^4 & \text{if } 0 \leq t \leq 1, \\
1 & \text{if } t > 1.
\end{cases}
\]

1. Compute the density of \( X \).

Let \( X_1, X_2, \ldots \) be i.i.d. random variables with the same law as \( X \). Let
\[ M_n = \min \{X_1, \ldots, X_n\} \]
be the smallest value among the \( n \) first \( X_k \)'s.

2. Find the cumulative distribution function of \( M_n \).

3. Let \( Z \) have cumulative distribution function
\[ F_Z(t) = 1 - \exp(-4t^3) \quad \text{for } t \geq 0 \] (and 0 otherwise). Prove that \( (n^{-1/3}M_n) \overset{\text{law}}{\rightarrow} Z \).

**Answers.**

1. Let \( 0 \leq t \leq 1 \), and let \( f \) be the density of \( X \);
\[ f(t) = F_X'(t) = 12t^2 - 12t^3 = 12t^2(1-t). \]

2. Let \( 0 \leq t \leq 1 \),
\[
\mathbb{P}(M_n \geq t) = \mathbb{P}(X_1 \geq t \cap X_2 \geq t \cap \cdots \cap X_n \geq t) = \mathbb{P}(X_1 \geq t)^n = (1 - 4t^3 - 3t^4)^n,
\]
so that
\[ F_{M_n}(t) = 1 - (1 - 4t^3 - 3t^4)^n \]
if \( 0 \leq t \leq 1 \), and 0 otherwise.

3. We use cumulative distribution functions.
\[
\mathbb{P}(n^{1/3}M_n \geq t) = 1 - \left( 1 - 4 \left( \frac{t}{n^{1/3}} \right)^3 - 3 \left( \frac{t}{n^{1/3}} \right)^4 \right)^n,
\]
\[
= 1 - \left( 1 - 4 \left( \frac{t}{n} \right)^3 + o(1/n) \right)^n,
\]
\[ \overset{n \to \infty}{\rightarrow} 1 - \exp(-4t^3) \]
(if \( t \geq 0 \)). Since \( t \mapsto 1 - \exp(-4t^3) \) is continuous, then this proves the convergence in distribution towards \( Z \).
**Exercise 5** [3pts]

Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables in $L^4$ (so that $X_i^2 \in L^2$). We assume that $E[X_i] = 0$ and $E[X_i^2] = 1$.

Set as usual $S_n = X_1 + X_2 + \cdots + X_n$.

1. Compute $E\left[ (S_{n+1})^2 \mid S_n \right]$.

2. Let $n < m$ be integers, prove that $\text{Cov}(S_n, S_m) = n$.

**Answers.**

1. 

$$
E\left[ (S_{n+1})^2 \mid S_n \right] = E\left[ (S_n + X_{n+1})^2 \mid S_n \right] = E[S_n^2 \mid S_n] + E[X_{n+1}^2 \mid S_n] + 2E[S_n X_{n+1} \mid S_n]
$$

$$
= S_n^2 + 1 + 2S_n E[X_{n+1}] = S_n^2 + 1.
$$

2. 

$$
E[S_n S_m] = E[S_n (S_m - S_n + S_n)] = E[S_n (S_m - S_n)] + E[S_n^2]
$$

$$
= E[S_n]E[(S_m - S_n)] + E[S_n^2] = 0 + n.
$$

Since $S_m - S_n = X_{n+1} + X_{n+2} + \cdots + X_m$ is independent from $S_n$. Now,

$$
\text{Cov}(S_n, S_m) = E[S_n S_m] - E[S_n]E[S_m] = n.
$$

**Exercise 6** [optional: 2pts]

Let $Z$ be a discrete random variable with values in $\mathbb{Z}_+ = \{0, 1, 2, \ldots \}$. Prove that

$$
Z \sim \text{Poisson}(\lambda) \iff \lambda E[h(Z + 1)] = E[Z h(Z)] \text{ for all bounded } h : \mathbb{Z}_+ \to \mathbb{R}.
$$

*(Hint: For the $\Leftarrow$ part, find a recursion formula for the sequence $p_k = P(Z = k)$.)

**Answers.**

We first prove that the "only if part". Assume that $Z$ follows the Poisson distribution with parameter $\lambda$, and take a bounded $\phi : \mathbb{Z} \to \mathbb{R}$.

$$
E[\phi(Z + 1)] = \sum_{k \geq 0} \phi(k + 1)e^{-\lambda} \frac{\lambda^k}{k!}
$$

$$
= \sum_{j \geq 1} \phi(j)e^{-\lambda} \frac{\lambda^{j-1}}{(j-1)!} \text{ (by change of variable } j \leftarrow k + 1) = \frac{1}{\lambda} \sum_{j \geq 1} \phi(j)e^{-\lambda} \frac{\lambda^j}{j!}
$$

$$
= \frac{1}{\lambda} \sum_{j \geq 0} \phi(j)e^{-\lambda} \frac{\lambda^j}{j!} \text{ (term } j = 0 \text{ cancels),}
$$
and the "only if part" is proved.
Assume now that $\lambda E[\phi(Z + 1)] = E[Z\phi(Z)]$ for all $\phi$. Apply this with 

$$\phi(z) = 1_{z=k},$$

we obtain

$$\lambda E[1_{Z+1=k}] = E[Z1_{Z=k}]$$

$$\lambda P(Z + 1 = k) = E[k1_{Z=k}]$$

$$\lambda P(Z + 1 = k) = kP(Z = k).$$

This is true for all $k \geq 1$, and then

$$p_k = \frac{\lambda}{k} p_{k-1} = \frac{\lambda}{k} \times \left( \frac{\lambda}{k-1} p_{k-2} \right) = \cdots = \frac{\lambda^k}{k!} p_0.$$

It remains to find $p_0$. We let you check that the only choice that makes $\sum_k p_k = 1$ is $p_0 = e^{-\lambda}$. 