Final Exam (duration: 3h)

The exam is open book (and notes). You may answer exercises in any order.

Exercise 1 [4pts]
Let (X, Y) have joint density $f(x, y) = 4\exp(-2y)\,\mathbb{1}_{y \geq x \geq 0}$.

1. Check that f is a density on $\mathbb{R}^2$.
2. Find the marginal densities f_X and f_Y of X and Y.
3. Are X and Y independent?
4. Find the density of $\dfrac{X}{Y}$.

Answers:

1. *f is obviously non-negative, let us compute*

$$\begin{aligned}\int\int f(x,y)dxdy &= \int_{x=0}^{+\infty}\left(\int_{y=0}^{\infty}\mathbb{1}_{y\geq x}4e^{-2y}dy\right)dx\\ &= \int_{x=0}^{+\infty}\left|-2e^{-2y}\right|_{y=x}^{\infty}dx\\ &= \int_{x=0}^{+\infty}2e^{-2x}dx = \left|-e^{-2x}\right|_{x=0}^{\infty} = 1.\end{aligned}$$

2. *By the previous computation*

$$f_X(x) = \int_{y=0}^{\infty}\mathbb{1}_{y\geq x}4e^{-2y}dy = 2e^{-2x}.$$

On the other hand

$$\begin{aligned}f_Y(y) &= \int_{x=0}^{+\infty}\mathbb{1}_{y\geq x}4e^{-2y}dx\\ &= 4e^{-2y}\int_{x=0}^{+\infty}\mathbb{1}_{y\geq x}\\ &= 4ye^{-2y}.\end{aligned}$$

3. *If X, Y were independent then*

$$f(x,y) = f_X(x)f_Y(y),$$

which is not the case.

4. *First note that with probability one $0 \leq X/Y \leq 1$. Let ϕ be a bounded and continuous function,*

$$\mathbb{E}[\phi(X/Y)] = \int_{y=0}^{+\infty}\left(\int_{x=0}^{y}\phi(x/y)4e^{-2y}dx\right)dy = \int_{y=0}^{+\infty}\left(\int_{x=0}^{y}\phi(x/y)dx\right)4e^{-2y}dy.$$

Make the change of variable $u \leftrightarrow x/y$ in the inner integral ($x \mapsto u(x)$ is increasing since $y > 0$) we have $du = dx/y$ and

$$\begin{aligned}\mathbb{E}[\phi(X/Y)] &= \int_{y=0}^{+\infty}\left(\int_{u=0}^{1}\phi(u)ydu\right)4e^{-2y}dy\\ &= \int_{u=0}^{1}\phi(u)\left(\int_{y=0}^{\infty}4ye^{-2y}dy\right)du = \int_{u=0}^{1}\phi(u)\times 1\times du.\end{aligned}$$

This proves that X/Y is uniform in $(0, 1)$.

Exercise 2 [2pts]

1. Let X be a non-negative random variable which has a density. Prove that for every $p \geq 1$

$$\mathbb{E}[X^p] = \int_0^{+\infty} pt^{p-1}\mathbb{P}(X \geq t)dt.$$

(note that both sides could be infinite).

Let X have cumulative distribution function $F_X(t) = \begin{cases} 1 - 1/t^6 & \text{if } t \geq 1, \\ 0 & \text{otherwise.} \end{cases}$.

2. Prove that $X \in L^p$ if and only if $p < 6$.

Answers:

1. Denote by f the density of X,

$$\begin{aligned}
\int_0^{+\infty} pt^{p-1}\mathbb{P}(X \geq t)dt &= \int_0^{+\infty} pt^{p-1} \left(\int_t^{+\infty} f(x)dx \right) dt \\
&= \int_{t=0}^{+\infty} pt^{p-1} \left(\int_{x=0}^{\infty} \mathbb{1}_{x \geq t} f(x)dx \right) dt \\
&= \int_{x=0}^{\infty} \int_{t=0}^{+\infty} pt^{p-1}\mathbb{1}_{x \geq t} f(x)dtdx \qquad \textit{(by Fubini n.1)} \\
&= \int_{x=0}^{\infty} f(x) \left(\int_{t=0}^{x} pt^{p-1}dt \right) dx \\
&= \int_{x=0}^{\infty} f(x)x^p dx = \mathbb{E}[X^p].
\end{aligned}$$

2. $X \geq 0$ so by the previous question

$$\mathbb{E}[X^p] = \int_0^{+\infty} pt^{p-1}t^{-6}dt = \int_0^{+\infty} pt^{p-7}dt$$

This integral is finite for every p such that $7 - p > 1$ i.e. $p < 6$.

Exercise 3 [3pts] Let $(X_n)_{n\geq 0}$ be the sequence of random variables defined by $X_0 = c$ and

$$X_{n+1} = aX_n + \varepsilon_{n+1} \qquad \text{(for every } n \geq 0)$$

where c, a are constants such that $|a| < 1$ and $(\varepsilon_n)_{n\geq 1}$ is a sequence of i.i.d. $\mathcal{N}(0, 1)$ (in particular ε_{n+1} is independent of X_n).

1. Prove that, for every $n \geq 1$, $X_n \sim \mathcal{N}(\mu_n, s_n^2)$ where

$$\mu_n = a^n c, \qquad s_n^2 = \frac{1 - a^{2n}}{1 - a^2}.$$

2. Give the characteristic function of X_n and find X such that $(X_n)_{n\geq 1} \overset{\text{(d)}}{\to} X$.

Answers:

1. For each n, aX_n and ε_{n+1} are two independent gaussian random variables, so that X_{n+1} is also gaussian. It remains to compute its mean and variance. Using induction assumptions:

$$\begin{aligned}
\mathbb{E}[X_{n+1}] &= \mathbb{E}[aX_n] + \mathbb{E}[\varepsilon_{n+1}] \\
&= a(a^n c) + 0. \\
\mathrm{Var}(X_{n+1}) &= \mathrm{Var}(aX_n) + \mathrm{Var}(\varepsilon_{n+1}) \qquad \textit{(by independence)} \\
&= a^2 \frac{1 - a^{2n}}{1 - a^2} + \frac{1 - a^2}{1 - a^2} \\
&= \frac{1 - a^{2n+2}}{1 - a^2}.
\end{aligned}$$

2. *By the course formula we have*

$$\Phi_{X_n}(t) = \exp\left(ita^n c - \frac{t^2}{2}\frac{1-a^{2n}}{1-a^2}\right) \overset{n\to\infty}{\to} \exp\left(it \times 0 - \frac{t^2}{2}\frac{1}{1-a^2}\right) = \Phi_X(t)$$

where $X \sim \mathcal{N}(0, 1/(1-a^2))$. *Therefore*

$$(X_n)_{n\geq 1} \overset{(d)}{\to} \mathcal{N}\left(0, \frac{1}{1-a^2}\right).$$

Exercise 4 [3pts]
Let $X_1, X_2, \dots$ defined on the same probability space and such that for every $n \geq 1$, X_n has density f_n, where f_n is defined by

$$f_n(x) = (n^2+1)x^{n^2}\mathbb{1}_{0\leq x\leq 1}.$$

1. Let $t \in (0,1)$, compute $\mathbb{P}(X_n \leq t)$. Prove that $(X_n)_{n\geq 1} \overset{\text{prob.}}{\to} 1$.
2. Find $\alpha > 0$ such that
$$n^\alpha(1-X_n) \overset{\text{(d)}}{\to} E,$$
where E follows the exponential distribution with parameter 1.

Answers:

1.
$$\mathbb{P}(X_n \leq t) = \int_0^t (n^2+1)x^{n^2}dx = \left|x^{n^2+1}\right|_{x=0}^t = t^{n^2+1}.$$
Let $\varepsilon > 0$, *since* $X_n \leq 1$
$$\mathbb{P}(|X_n - 1| \geq \varepsilon) = \mathbb{P}(X_n \leq 1-\varepsilon) = (1-\varepsilon)^{n^2+1} \to 0.$$
Therefore $(X_n)_{n\geq 1} \overset{prob.}{\to} 1$.

2. *We compute the cdf of* $n^\alpha(1-X_n)$. *Let* $t \geq 0$ *and* n *large enough (such that* $n^\alpha \geq t$ *below):*
$$\mathbb{P}\left(n^\alpha(1-X_n) \leq t\right) = \mathbb{P}\left(X_n \geq 1 - n^{-\alpha}t\right) = 1 - \left(1 - \frac{t}{n^\alpha}\right)^{n^2+1} \to 1 - e^{-t},$$
if $\alpha = 2$. *Since* $F_E(t) = 1 - e^{-t}$ *and since* F_E *is continuous, this proves that* $n^2(1-X_n) \overset{(d)}{\to} E$.

Exercise 5 [3pts] Let $p \in (0,1)$ and $X_1, X_2, \dots$ be a sequence of i.i.d. random variables such that

$$\mathbb{P}(X_n = 1) = p, \qquad \mathbb{P}(X_n = 0) = 1-p$$

and set $S_n = X_1 + \cdots + X_n$ (and $S_0 = 0$).

1. Compute $\mathbb{E}\left[\left(\frac{p+1}{p}\right)^{X_n}\right]$.
2. Prove that for every $n \geq 0$
$$\mathbb{E}\left[\left(\frac{p+1}{p}\right)^{S_{n+1}}\middle|S_n\right] = 2\left(\frac{p+1}{p}\right)^{S_n}.$$

Answers:

1.
$$\mathbb{E}\left[(1+1/p)^{X_n}\right] = \left((1+1/p)^1 p + (1+1/p)^0(1-p)\right) = 2.$$

2.

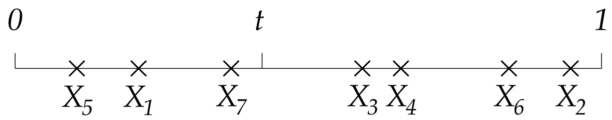
$$\begin{aligned}
\mathbb{E}\left[(1+1/p)^{S_{n+1}}|S_n\right] &= \mathbb{E}\left[(1+1/p)^{S_n+X_{n+1}}|S_n\right] \\
&= \mathbb{E}\left[(1+1/p)^{S_n}(1+1/p)^{X_{n+1}}|S_n\right] \\
&= (1+1/p)^{S_n}\mathbb{E}\left[(1+1/p)^{X_{n+1}}|S_n\right] \qquad \textit{(S_n is measurable w.r.t to S_n)} \\
&= (1+1/p)^{S_n}\mathbb{E}\left[(1+1/p)^{X_{n+1}}\right] \qquad \textit{(X_{n+1} is independent of S_n)} \\
&= (1+1/p)^{S_n}\mathbb{E} \times 2.
\end{aligned}$$

Problem. [5pts] The aim of this problem is to introduce the *Poisson point process* which is used to model random "points" in time and space, such as the arrival times of customers at a service center. We will study some of its properties in questions 1-2 and in questions 3-4 we will show a connection with the exponential distribution.

The *Poisson point process* on the interval $[0,1]$ is the random set of points defined in the following way:

- Pick N at random according to the Poisson distribution with parameter 1.
- Then, conditional on N, pick at random N i.i.d. points $X_1, \dots X_N$ with the uniform distribution in $[0,1]$.

Here is an example if $N = 7$:



For any $t \in (0,1)$, set

$$\begin{cases} P(t) = \text{card}\{i \text{ such that } X_i \in (0,t)\} \\ Q(t) = \text{card}\{i \text{ such that } X_i \in (t,1)\} \end{cases}$$

In the example above $P(t) = 3, Q(t) = 4$.

1. Prove that

$$\mathbb{P}(P(t) = i, Q(t) = j | N = n) = \binom{n}{i} t^i (1-t)^j \mathbb{1}_{\{i+j=n\}}.$$

2. Deduce from the above that $P(t)$ and $Q(t)$ are independent random variables such that

$$P(t) \sim \text{Poisson}(t), \qquad Q(t) \sim \text{Poisson}(1-t).$$

3. Set

$$\mathcal{M} = \begin{cases} \min\{X_1, \dots, X_N\} & \text{if } N \geq 1, \\ 1 & \text{if } N = 0. \end{cases}$$

($\mathcal{M}$ is the left-most point of the Poisson point process, and we set $\mathcal{M} = 1$ if no point is picked. In the example above $\mathcal{M} = X_5$.)
Using the previous question, compute $\mathbb{P}(\mathcal{M} \geq t)$ for all $t \in (0,1)$.

4. Let E have the exponential distribution with parameter one. Set $Z = \min\{E, 1\}$, *i.e.*

$$Z = \begin{cases} E & \text{if } E < 1, \\ 1 & \text{if } E \geq 1. \end{cases}$$

Prove that $\mathcal{M}$ has the same distribution as Z.

Answers:

1. *Conditional on* $N = n$*, one picks* n *i.i.d. uniform r.v. Each of them falls in* $(0, t)$ *independently with probability* t*, so* $P(t) \sim \text{Binom}(n, t)$ *and* $Q(t)$ *is just* $n - P(t)$*. So for* $i + j = n$

$$\begin{aligned}\mathbb{P}\left(P(t) = i, Q(t) = j | N = n\right) &= \mathbb{P}\left(P(t) = i | N = n\right) \\ &= \mathbb{P}\left(\text{Binom}(n, t) = i\right) \\ &= \binom{n}{i} t^i (1-t)^{n-i} \\ &= \binom{n}{i} t^i (1-t)^j.\end{aligned}$$

2. *For every* $i, j \geq 0$

$$\begin{aligned}\mathbb{P}\left(P(t) = i, Q(t) = j\right) &= \sum_{n=0}^{+\infty} \mathbb{P}\left(P(t) = i, Q(t) = j, N = n\right) \\ &= \sum_{n=0}^{+\infty} \mathbb{P}\left(P(t) = i, Q(t) = j | N = n\right) \mathbb{P}\left(N = n\right) \\ &= \sum_{n=0}^{+\infty} \binom{n}{i} t^i (1-t)^j \mathbb{1}_{\{i+j=n\}} \frac{e^{-1}}{n!} \\ &= \binom{i+j}{i} t^i (1-t)^j \frac{e^{-1}}{(i+j)!} \\ &= \frac{(i+j)!}{i!j!} t^i (1-t)^j \frac{e^{-t} e^{-(1-t)}}{(i+j)!} \\ &= \frac{t^i e^{-t}}{i!} \frac{(1-t)^j e^{-(1-t)}}{j!}.\end{aligned}$$

3. *Because of the above, for* $t < 1$*,* $\mathbb{P}(\mathcal{M} \geq t) = \mathbb{P}(P(t) = 0) = e^{-t}$*.*

4. *Let* $t \in (0, 1)$*, then* $\mathbb{P}(Z \geq t) = \mathbb{P}(\min E, 1 \geq t) = \mathbb{P}(E \geq t) = e^{-t}$*. It remains to prove that* $\mathbb{P}(Z = 1) = \mathbb{P}(\mathcal{M} = 1)$*.*

$$\mathbb{P}(Z = 1) = \mathbb{P}(E \geq 1) = e^{-1}.$$

and

$$\mathbb{P}(\mathcal{M} = 1) = \mathbb{P}(N = 0) = \mathbb{P}(P(1) = 0) = \frac{1^0}{0!} e^{-1} = e^{-1}.$$