MINI-COURSE: RANDOM UNIFORM PERMUTATIONS

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This course is at the interplay between Probability and Combinatorics. It is intended for Master students with a background in Probability (random variables, expectation, conditional probability).

The question we will adress is "What can we say about a *typical* large permutation?": the number of cycles, their lengths, the number of fixed points,... This is also a pretext to present some universal phenomena in Probability: reinforcement, the Poisson paradigm, size-bias,...

Contents

1 Brief reminder on permutations

Before we turn to *random* permutations, we will give a few definitions regarding non-random (or *deterministic* permutations).

A *permutation* of size $n \geq 1$ is a bijection $\sigma : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$. For example

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 4 & 3 & 1 \end{array}$$

is a permutation of size 4. In these notes we often write a permutation with its one-line representation $\sigma(1)\sigma(2)\dots\sigma(n)$. For example the above permutation is simply written 2431.

There are $n!$ permutations of size n.

Cycle decomposition

For our purpose, there is a convenient alternative way to encode a permutation: by its *cycle decomposition.* A *cycle* is a finite sequence of distinct integers, defined up to the cycle order. This means that the three following denote the same cycle:

$$(8, 3, 4) = (3, 4, 8) = (4, 8, 3),$$

while $(8, 3, 4) \neq (8, 4, 3)$.

The *cycle decomposition* of a permutation σ is defined as follows. We give the theoretical algorithm and detail the example of

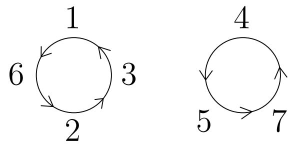
$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 6 & 3 & 1 & 5 & 7 & 2 & 4 \end{array}$$

Algorithm	**Example**
Start with 1st cycle (1)	(1)
Add to this cycle $\sigma(1)$, then $\sigma(\sigma(1))$, then $\sigma(\sigma(\sigma(1)))$, and so one until one of this number is one.	$(1) \to (1,6) \to (1,6,2) \to (1,6,2,3)$ and the cycle is over since $\sigma(3) = 1$.
Start the 2d cycle with a number which has not been seen before.	1st cycle $(1,6,2,3)$. 2d cycle: (4)
Complete the 2d cycle with same procedure.	1st cycle $(1,6,2,3)$. 2d cycle: $(4) \to (4,5) \to (4,5,7)$.
Create new cycles until there is no remaining number.	Done.

Finally, the cycle decomposition of σ is

$$(1,6,2,3),(4,5,7)$$

It is convenient to represent the cycle decomposition of σ with the following diagram:



Exercise 1 What is the cycle decomposition of 62784315?

2 How to simulate a random uniform permutation?

We will first discuss the following question. Imagine that you are given a random number generator `rand` (in your favourite programming language) which returns independent uniform random variables. How to use `rand` to simulate a random uniform permutation of size n?

2.1 The naive algorithm

It works as follows:

- Pick $\sigma(1)$ uniformly at random in $\{1,2,\dots,n\}$ (n choices);
- Pick $\sigma(2)$ uniformly at random in $\{1,2,\dots,n\} \setminus \{\sigma(1)\}$ ($n-1$ choices);
- Pick $\sigma(3)$ uniformly at random in $\{1,2,\dots,n\} \setminus \{\sigma(1),\sigma(2)\}$ ($n-2$ choices),

and so on until $\sigma(n)$ (1 choice).

By construction every permutation occurs with probability $1/n!$ so the output is uniform.

2.2 The "continuous" algorithm

- Pick continuous random variables $X_1, X_2, \ldots, X_n$, independently and uniformly in $(0,1)$;
- With probability 1 the n values are pairwise distinct. Therefore there exists a unique permutation σ such that
$$X_{\sigma(1)} < X_{\sigma(2)} < X_{\sigma(3)} < \cdots < X_{\sigma(n)}.$$
- This σ is your output.

Proposition 1. *For every n, the output of the continuous algorithm is uniform among the $n!$ permutations of size n.*

Proof. It is not obvious that σ is uniform in this case.
Step 1: The n values are distinct. We have to prove that

$$\mathbb{P}\left(\text{ for all } i \neq j,\ X_i \neq X_j \right) = 1.$$

We prove that the complement event $\{$there are i, j such that $X_i = X_j\}$ has probability zero. First we notice that

$$\begin{aligned}\mathbb{P}(\text{ there are } i \neq j \text{ such that } X_i = X_j) &= \mathbb{P}\left(\cup_{i\neq j}\{X_i = X_j\}\right)\\ &\leq \sum_{i\neq j} \mathbb{P}\left(X_i = X_j\right),\end{aligned}$$

by the union bound[(i)]. Now,

$$\mathbb{P}\left(X_i = X_j\right) = \int_{(0,1)^2} \mathbf{1}_{x=y} dx dy = \int_{y\in(0,1)} \left(\int_{x\in(0,1)} \mathbf{1}_{x=y} dx \right) dy = \int_{y\in(0,1)} \left(\int_{x=y}^{y} dx \right) dy = \int_{y\in(0,1} 0 \times dy = 0.$$

Step 2: The output σ is uniform. To avoid messy notations we make the proof in the case $n = 3$. Since the 3 values X_1, X_2, X_3 are distinct we have

$$\begin{aligned} 1 &= \mathbb{P}(X_1 < X_2 < X_3) + \mathbb{P}(X_1 < X_3 < X_2) + \mathbb{P}(X_2 < X_1 < X_3)\\ &\quad + \mathbb{P}(X_2 < X_3 < X_1) + \mathbb{P}(X_3 < X_1 < X_2) + \mathbb{P}(X_3 < X_2 < X_1)\\ &= \int_{(0,1)^3} \mathbf{1}_{x_1<x_2<x_3} dx_1 dx_2 dx_3 + \int_{(0,1)^3} \mathbf{1}_{x_1<x_3<x_2} dx_1 dx_2 dx_3 + \int_{(0,1)^3} \mathbf{1}_{x_2<x_1<x_3} dx_1 dx_2 dx_3\\ &\quad + \int_{(0,1)^3} \mathbf{1}_{x_2<x_3<x_1} dx_1 dx_2 dx_3 + \int_{(0,1)^3} \mathbf{1}_{x_3<x_1<x_2} dx_1 dx_2 dx_3 + \int_{(0,1)^3} \mathbf{1}_{x_3<x_2<x_1} dx_1 dx_2 dx_3. \end{aligned}$$

Now, x_1, x_2, x_3 are dummy variables in the above integrals, so they are interchangeable. Therefore, these 6 integrals are identical and each of these is $1/6 = 1/3!$. ☐

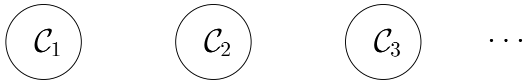
2.3 The "Chinese restaurant" algorithm

We introduce the Chinese restaurant algorithm, also called the Fisher-Yates algorithm (or even Fisher-Yates-Knuth algorithm). The main difference with the two previous algorithms is that the output σ will be described through its cycle decomposition.

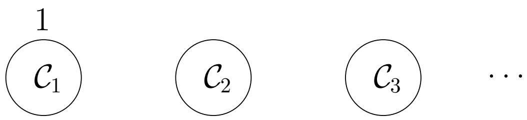
The algorithm runs as follows:

[(i)] The union bound says that $\mathbb{P}\left(\bigcup_{n\geq 1} A_n\right) \leq \sum_{n\geq 1} \mathbb{P}(A_n)$ for every sequence of events (A_n).

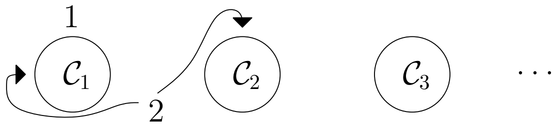
- Assume we are given infinitely many ”restaurant tables” $\mathcal{C}_1, \mathcal{C}_2, \ldots$. These tables are large enough so that an arbitrary number of people can sit at each table.



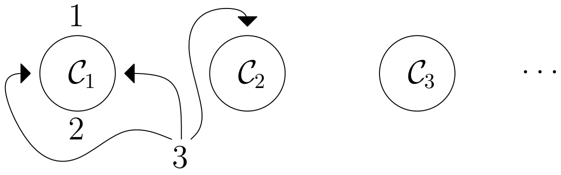
- Infinitely many customers $1, 2, 3, \ldots$ enter the restaurant, one at a time. Put Customer n.1 at table $\mathcal{C}_1$:



- With equal probability one-half, put Customer n.2 either at the same table as 1 (on its right) or alone at the new table $\mathcal{C}_2$:

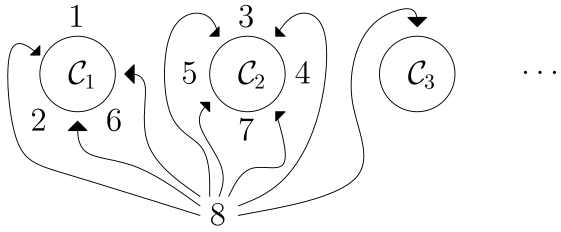


- With equal probability one-third, put Customer n.3 either on the right of 1, or on the right of 2, or alone at the first empty table:

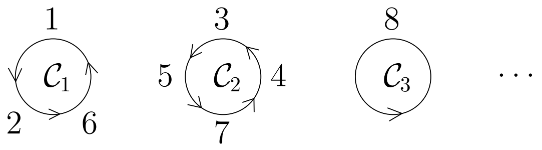


- $\ldots$

- Assume that customers $1, 2, \ldots, n-1$ are already installed. With equal probability $1/n$, put Customer n either on the right of 1, ..., or on the right of $n-1$, or alone at the first empty table (here $n = 8$):



Now, we return the permutation σ whose cycle decomposition corresponds to table repartitions. Assume here that 8 sits alone, we obtain the diagram



This can also be written (126)(3547)(8). The corresponding permutation is

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 6 & 5 & 3 & 7 & 1 & 4 & 8 \end{array}$$

$\boxed{\textbf{Exercise 2}}$ Take $n = 4$. What is the probability that the output of the algorithm is the permutation 4231? (Hint: First write the cycle decomposition of 4231.)

Proposition 2. *For every n, the output of the Chinese restaurant algorithm is uniform among the $n!$ permutations of size n.*

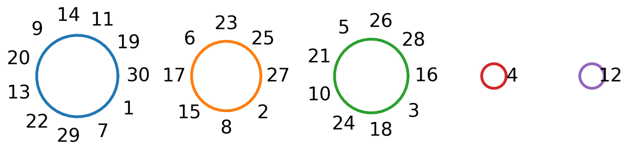
Proof. By construction, each table repartition with n customers occurs with the same probability

$$1 \times \frac{1}{2} \times \frac{1}{3} \times \cdots \times \frac{1}{n}.$$

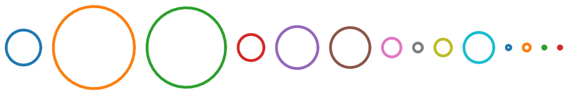
Now, each table repartition corresponds to exactly one permutation of size n. Therefore each permutation occurs with probability $1/n!$. ☐

Simulations

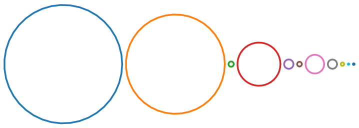
Here is a simulation for $n = 30$:



Here is a simulation for $n = 2000$ (We only represent sizes of tables. They have respective sizes $122, 673, 631, 68, 176, 159, 35, 8, 28, 91, 2, 5, 1, 1.$):



A last simulation for $n = 30000$. Tables have sizes $15974, 11238, 31, 2121, 99, 25, 397, 97, 13, 2, 3.$



For more on the Chinese restaurant we refer to [5]. On the following webpage you can run simulations of the Chinese restaurant by yourself:

`http://gerin.perso.math.cnrs.fr/ChineseRestaurant.html`

3 Typical properties of a random uniform permutation

Frow now on S_n denotes a random uniform permutation of size n, generated by any of the previous algorithms.

3.1 Number of fixed points

Definition 1. *Let σ be a permutation of size n. The integer $1 \leq i \leq n$ is a fixed point of σ if $\sigma(i) = i$.*

For example, 2431 has a unique fixed point at $i = 3$.

Proposition 3. *Let F_n be the number of fixed points of S_n. For every n, we have that $\mathbb{E}[F_n] = 1$.*

Proof. We write $F_n = \sum_{i=1}^{n} X_i$, where

$$X_i = \begin{cases} 1 & \text{if } S_n(i) = i, \\ 0 & \text{otherwise} \end{cases}.$$

By linearity of expectation we have that

$$\mathbb{E}[F_n] = \mathbb{E}[X_1 + \cdots + X_n] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n],$$

and we are left to compute $\mathbb{E}[X_i]$ for every i. Now,

$$\mathbb{P}(X_i = 1) = \mathbb{P}(S_n(i) = i) = \frac{\text{card}\,\{\text{permutations } s \text{ of size } n \text{ with } s(i) = i\}}{\text{card}\,\{\text{permutations of size } n\}} = \frac{(n-1)!}{n!} = \frac{1}{n}.$$

(For the last inequality we notice that a permutation s of size n with $s(i) = i$ is in fact a permutation of size $n - 1$.)
Therefore we have that

$$\mathbb{E}[X_i] = 1 \times \mathbb{P}(X_i = 1) + 0 \times \mathbb{P}(X_i = 0) = 1/n.$$

Finally

$$\mathbb{E}[F_n] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n] = n \times 1/n = 1.$$

$\square$

Remark . *With a little more work one can compute* $\mathrm{Var}(F_n)$. *It suffices to compute* $\mathrm{Cov}(X_i, X_j)$ *and use formula* $\mathrm{Var}(\sum X_i) = \sum_i \mathrm{Var}(X_i) + \sum_{i \neq j} \mathrm{Cov}(X_i, X_j)$. *One finds that for every* n

$$\mathrm{Var}(F_n) = 1 - \frac{1}{n} + \frac{1}{n^2(n-1)}.$$

The Poisson paradigm

There is a general phenomenon in probability known as the *Poisson paradigm.* It says that if X_i's are 0/1 random variable such that

1. $\mathbb{E}[X_i] = \mathbb{P}(X_i = 1)$ is "small" for every i ;
2. X_i's are "almost" independent ;

then $X = \sum X_i$ is almost distributed like the Poisson distribution with mean $\sum \mathbb{E}[X_i]$. Here $\sum \mathbb{E}[X_i] = \sum_{i=1}^{n} 1/n = 1$ and one can make the Poisson paradigm rigorous:

Proposition 4. *Let* $(S_n)_n$ *be a sequence of random uniform permutations, and let* F_n *be the number of fixed points of* S_n. *Then* F_n *converges* in distribution *to the Poisson distribution with mean* 1, i.e.

$$\mathbb{P}(F_n = k) \overset{n\to+\infty}{\to} \mathbb{P}(\mathrm{Poisson}(1) = k) = \frac{e^{-1}}{k!},$$

for every $k = 0, 1, 2, \ldots$.

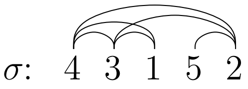
A combinatorial proof can be found at [8]. For more on the Poisson paradigm, we refer to [2].

3.2 Number of inversions

An *inversion* in σ is a pair (i, j) such that

$$\begin{cases} i < j, \\ \sigma(i) > \sigma(j) \end{cases}.$$

Let $\mathrm{Inv}(\sigma)$ be the number of inversions of σ. For example, if $\sigma = 43152$ then $\mathrm{Inv}(\sigma) = 6$ (each arc counts for an inversion):



Proposition 5. *For every* n, *let* S_n *be a uniform random permutation of size* n. *Then*

$$\mathbb{E}[\mathrm{Inv}_n(S_n)] = \frac{n(n-1)}{4}.$$

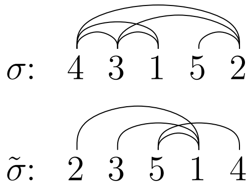
Proof. We will make a combinatorial proof, without any computation. First, let $\tilde{\sigma}$ be the *reversed* permutation of σ: for every $1 \leq i \leq n$,

$$\tilde{\sigma}(i) = n + 1 - \sigma(i).$$

For instance, if $\sigma = 43152$ then $\tilde{\sigma} = 23514$. Then by construction we have that an arbitrary pair (i, j) is an inversion for σ if and only if it is not an inversion for $\tilde{\sigma}$. We deduce that

$$\mathrm{Inv}(\sigma) + \mathrm{Inv}(\tilde{\sigma}) = \mathrm{card}\,\{\text{ all pairs } 1 \leq i < j \leq n\} = \binom{n}{2} = \frac{n(n-1)}{2}.$$

Here we see that $\mathrm{Inv}(43152) + \mathrm{Inv}(23514) = 6 + 4 = \binom{5}{2}$:



Now, we apply the above equality to $\sigma = S_n$ and take expectations of both sides:

$$\mathbb{E}\big[\mathrm{Inv}(S_n)\big] + \mathbb{E}\big[\mathrm{Inv}(\tilde{S_n})\big] = \frac{n(n-1)}{2}.$$

But now, it is obvious that $\sigma \mapsto \tilde{\sigma}$ is a bijection so it preserves the uniform measure. Therefore $\tilde{S_n}$ is also a uniform random permutation and we have $\mathbb{E}\big[\mathrm{Inv}(S_n)\big] = \mathbb{E}\big[\mathrm{Inv}(\tilde{S_n})\big]$. The proof is done. □

3.3 Number of cycles

Proposition 6. *Let C_n be the number of cycles of S_n. When $n \to +\infty$,*

$$\mathbb{E}[C_n] \overset{n\to+\infty}{\sim} \log(n).$$

Proof. We may assume that S_n is the output of the Chinese restaurant algorithm. All along the process of the Chinese restaurant, a new cycle appears when a customer sits at a new table:

$$C_n = \sum_{i=1}^{n} Z_i,$$

where

$$Z_i = \begin{cases} 1 & \text{if Customer } i \text{ sits at a new table,} \\ 0 & \text{otherwise} \end{cases}.$$

Customer i sits at a new table with probability $1/i$, therefore $\mathbb{E}\left[Z_i\right] = 1/i$. Then,

$$\mathbb{E}[C_n] = \mathbb{E}\left[\sum_{i=1}^{n} Z_i\right] = \sum_{i=1}^{n} \mathbb{E}\left[Z_i\right] = \sum_{i=1}^{n} \frac{1}{i}.$$

Now, we use the fact that(ii) $\sum_{i=1}^{n} \frac{1}{i} \sim \log(n)$. □

(ii) See `https://en.wikipedia.org/wiki/Harmonic_series_(mathematics)`

3.4 Size of the first cycle/first table

Let $T_1(n)$ be the number of customers at Table 1 in the Chinese restaurant process at time n. By Proposition 2, we have that the random variable $T_1(n)$ has the distribution of the cycle of 1 in the cycle decomposition of a random uniform permutation of size n.

Proposition 7. *For every n, the random variable $T_1(n)$ is uniformly distributed in $\{1, 2, \ldots, n\}$,* i.e.

$$\mathbb{P}(T_1(n) = i) = \frac{1}{n}, \qquad \text{for every } i \in \{1, 2, \ldots, n\}\,.$$

Remark . 1. *The distribution of the sequence $(T_1(n))_{n\geq 1}$ is actually known as the* Pólya Urn process *[6].*

2. *A nice problem related to Proposition 7 is given by the* 100 prisoners problem *[7].*

Proof. **1st proof: Probability.**
The proof goes by induction. For $n = 1$ this is obvious since with probability one $T_1(1) = 1$.

Assume now that for some $n \geq 1$, the random variable $T_1(n)$ is uniform in $\{1, 2, \ldots, n\}$. If $T_1(n) = i$, then Customer $n + 1$ sits at table 1 with probability $i/(n + 1)$.

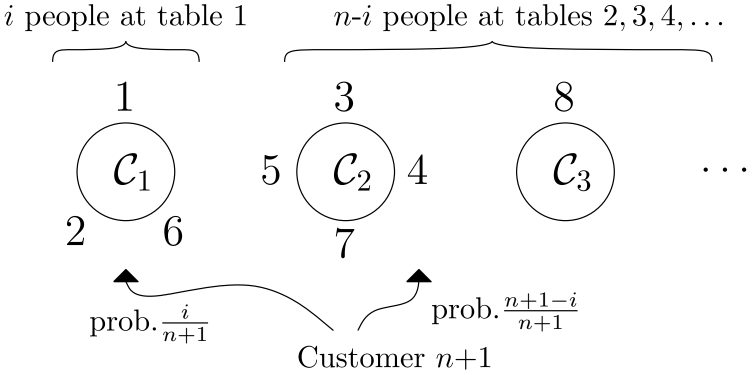


Figure: *A sketch of the situation when Customer $n + 1$ tries to sit.*

Therefore

$$T_1(n+1) = \begin{cases} i+1 & \text{with probab. } \frac{i}{n+1}, \\ i & \text{with probab. } \frac{n+1-i}{n+1}. \end{cases} \tag{1}$$

Fix $j \in \{1, \ldots, n + 1\}$. The above argument implies that

$$\begin{aligned}
\mathbb{P}(T_1(n+1) = j) &= \mathbb{P}(T_1(n+1) = j \cap T_1(n) = j) + \mathbb{P}(T_1(n+1) = j \cap T_1(n) = j-1) \\
&= \mathbb{P}(T_1(n+1) = j | T_1(n) = j)\mathbb{P}(T_1(n) = j) \\
&+ \mathbb{P}(T_1(n+1) = j | T_1(n) = j-1)\mathbb{P}(T_1(n) = j-1) \\
&= \frac{n+1-j}{n+1} \times \mathbb{P}(T_1(n) = j) \qquad \text{(apply (1) with } i = j.\text{)} \\
&+ \frac{j-1}{n+1} \times \mathbb{P}(T_1(n) = j-1) \qquad \text{(apply (1) with } i = j-1.\text{)} \\
&= \frac{n+1-j}{n+1} \times \frac{1}{n} + \frac{j-1}{n+1} \times \frac{1}{n} \qquad \text{(recall } T_1(n) \text{ is uniform)} \\
&= \frac{n}{(n+1)n} = \frac{1}{n+1},
\end{aligned}$$

which proves that $T_1(n+1)$ is uniform in $\{1,\dots,n+1\}$.

2d proof: Combinatorics.

For $i=1,\dots,n$, let us enumerate the permutations in which $T_1(n)=i$. We have to choose $i-1$ elements $x_1,\dots,x_{i-1}$ which belong to this cycle, and put them in a given order. Then, the $n-i$ remaining elements form a permutation of size $n-i$. Therefore

$$\begin{aligned}\mathbb{P}(T_1(n)=i) &= \frac{\mathrm{card}\,\{\text{permutations of size } n \text{ with } T_1(n)=i\}}{n!}\\ &= \frac{1}{n!}\binom{n-1}{i-1}(i-1)!(n-i)!\\ &= \frac{1}{n!}\frac{(n-1)!}{(i-1)!(n-i)!}(i-1)!(n-i)! = \frac{1}{n}.\end{aligned}$$

□

Discussion: the reinforcement phenomenon

The Chinese restaurant process illustrates the *reinforcement phenomenon* which is very common in Probability. It is also known as the "rich gets richer" phenomenon. Indeed, we observe that the more people there are at Table 1 at a given time, the more there will be in the future.

As an application, it turns out that because Table 1 appears sooner than Table 2, Table 1 is much more occupied (in average) than Table 2.

Proposition 8. *For large n, we have that*

$$\mathbb{E}[T_1(n)] \overset{n\to+\infty}{\sim} \frac{n}{2}, \qquad \mathbb{E}[T_2(n)] \overset{n\to+\infty}{\sim} \frac{n}{4}.$$

Proof. First, we claim that conditionally on the event $\{T_1(n)=i\}$, then $T_2(n)$ is uniformly distributed in $\{1,2,\dots,n-i\}$: for every $j\leq n-i$ we have

$$\mathbb{P}(T_2(n)=j \mid T_1(n)=i) = \begin{cases}\frac{1}{n-i} & \text{if } i<n,\\ 0 & \text{if } i=n.\end{cases}$$

We skip the proof, which is very similar to the proof of Proposition 7 (in this case the combinatorial proof is easier).

Consequently, if we condition on the event $\{T_1(n)=i\}$ we have that

$$\begin{aligned}\mathbb{E}[T_2(n)|T_1(n)] &= \mathbb{E}[\text{ Uniform random var. in } \{1,2,\dots,n-T_1(n)\}]\\ &= \frac{1+n-T_1(n)}{2}.\end{aligned}$$

Now, by the tower property of conditional expectation[iii] we obtain

$$\mathbb{E}[T_2(n)] = \mathbb{E}\Big[\mathbb{E}[T_2(n)|T_1(n)]\Big] = \mathbb{E}\left[\frac{1+n-T_1(n)}{2}\right] = \frac{1+n-n/2}{2} \sim \frac{n}{4}.$$

□

[iii] This says that $\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X]$.

Discussion: the size-bias phenomenon

We conclude by investigating an apparent paradox:

- In average, there are $n/2$ people at the same table as 1. But recall that the output of the Chinese restaurant process is uniform in $\mathfrak{S}_n$ so by symmetry, every element in $\{1, 2, \dots, n\}$ plays the same role: this table can be considered as a *typical* table.
- There are in average $\log(n)$ distinct tables, so a *typical* table should have (in average) about

  $$\frac{\text{Number of customers}}{\text{Number of tables}} \approx \frac{n}{\log(n)} \ll \frac{n}{2}$$

  customers.

The paradox is that Table 1 is *not* typical: by saying that 1 sits at this table the size of this table is biased. The size of Table 1 is overestimated compared to a "true" typical table. This is the *size-bias* phenomenon, whose a very nice introduction can be found in [1].

4 How to sort S_n efficiently: average-case analysis of `Quicksort`

We will discuss a different topic regarding random permutations: the analysis of sorting algorithms. We will focus on one of the most famous: `Quicksort`.

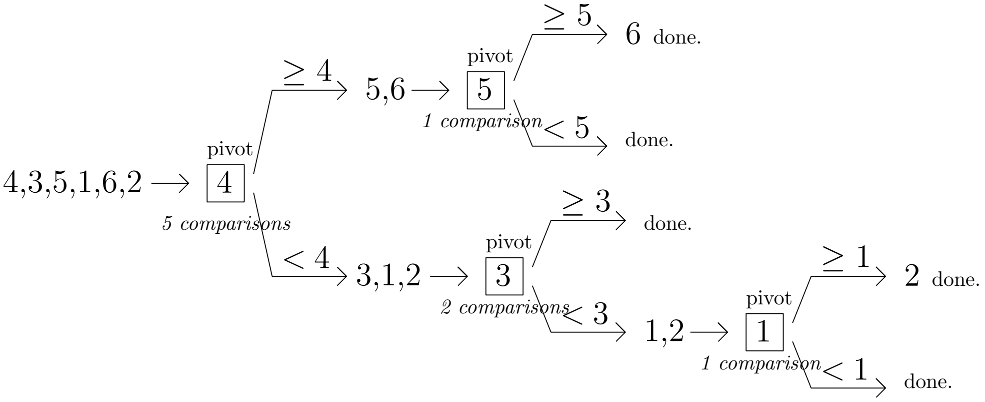
4.1 The algorithm

Input: Sequence of numbers $x_1, x_2, \dots, x_n$
Output: Re-ordered sequence $x_{\sigma(1)} \leq x_{\sigma(2)} \cdots \leq x_{\sigma(n)}$

The algorithm uses the *Divide-and-Conquer* strategy, there are three steps:

1. Call x_1 the *pivot* of the list.
2. Compare all the elements $x_2, \dots, x_n$ with x_1 and re-order the list so that
   (a) elements $< x_1$ come before the pivot,
   (b) elements $\geq x_1$ come after the pivot.
3. Recursively apply strategy to both sub-lists.

Here are the first steps applied to the permutation 435162:



4.2 Average-case analysis

We consider that the cost of the algorithm driven on $x_1, \dots, x_n$ is given by the number $\mathrm{Comp}(x_1, \dots, x_n)$ of pairwise comparisons between x_i's. For instance, in the above example we have that

$$\mathrm{Comp}(4,3,5,1,6,2) = 5+1+2+1 = 9.$$

If the input is random, then Comp is a random variable.

Proposition 9. *Let $X_1, \dots, X_n$ be independent random variables uniform in the interval $(0,1)$. Then, when $n \to +\infty$,*

$$\mathbb{E}\big[\mathrm{Comp}(X_1, \dots, X_n)\big] = 2n\log(n) + \mathrm{o}(n\log(n)).$$

Both the algorithm and its analysis were provided by Hoare [4]. A modern reference is [3].

Proof. By construction X_1 is the first pivot. Denote by $Y_1, \dots, Y_{I-1}$ be the numbers $> X_1$, and $Z_1, \dots, Z_{n-I}$, so that I is the (random) rank of X_1 in the sequence. Because of the recursive strategy the number of comparisons is given by

$$\mathrm{Comp}(X_1, \dots, X_n) = \underbrace{n-1}_{\text{Comp. with } X_1} + \mathrm{Comp}(Y_1, \dots, Y_{I-1}) + \mathrm{Comp}(Z_1, \dots, Z_{n-I}). \qquad (\star)$$

We omit the proofs of the two following claims:

- The rank I is uniform in $1, 2, \dots, n$.
- Conditionally on X_1, the Y_j's are i.i.d. (and uniform in $(0, X_1)$) and the Z_j's are i.i.d. (and uniform in $(X_1, 1)$).

Therefore, if we take expectations of both sides of $(\star)$ and put $c_n = \mathbb{E}\big[\mathrm{Comp}(X_1, \dots, X_n)\big]$ then we obtain

$$c_n = n-1+\sum_{i=1}^{n} \mathbb{P}(I=i)c_{i-1} + c_{n-i} = n-1+\frac{1}{n}\sum_{i=1}^{n} c_{i-1} + c_{n-i} = n-1+\frac{2}{n}\sum_{i=1}^{n} c_{i-1},$$

with $c_0 = c_1 = 0$. We easily check that c_n's satisfy

$$nc_n = 2n-2+(n+1)c_{n-1},$$

which rewrites as:

$$\frac{c_n+2n}{n+1} = \frac{2}{(n+1)} + \frac{c_{n-1}+2(n-1)}{n}.$$

If we put $d_n := \frac{c_n+2n}{n+1}$ we have that

$$d_n = \frac{2}{n+1} + \frac{2}{n} + \frac{2}{n-1} + \cdots + \frac{2}{5} + \frac{2}{4} + d_2.$$

i.e. $d_n = 2\log(n) + \mathrm{o}(\log(n))$. Finally,

$$c_n = 2n\log(n) + \mathrm{o}(n\log(n)).$$

□

(We observe that the number of comparisons $\mathrm{Comp}(X_1, \dots, X_n)$ *only depends on the relative order of the X_i's, not on their exact values. Therefore Proposition 9 remains true (with the same proof) if X_i's are i.i.d. with an arbitrary density.)*

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