Bachelor of Ecole Polytechnique
Computational Mathematics, year 2, semester 1
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Graphs and Probability 1: Adjacency Matrices

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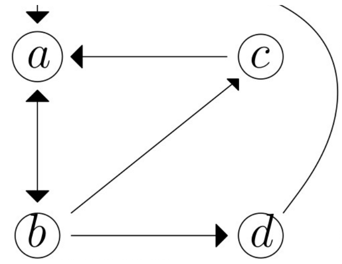
```python
# execute this part to modify the css style
from IPython.core.display import HTML
def css_styling():
    styles = open("./style/custom2.css").read()
    return HTML(styles)
css_styling()
```

```python
## loading python libraries

# necessary to display plots inline:
%matplotlib inline

# load the libraries
import matplotlib.pyplot as plt # 2D plotting library
import numpy as np              # package for scientific computing

from math import *              # package for mathematics (pi, arctan, sqrt, factorial
import sympy as sympy             # package for symbolic computation
from sympy import *
```

Adjacency Matrices

Exercise 1. A first example

Let G be the following graph:



Do it yourself. Use the adjacency matrix of G to compute the number of paths of length 20 from a to b in G.

Answers.

Enumeration of words

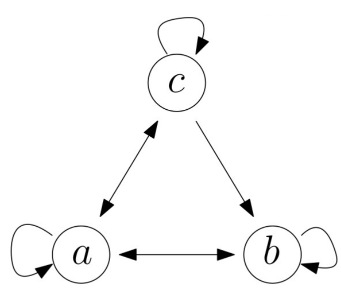
Exercise 2. b never followed by c

Do it yourself. We consider words w with letters a, b, c. Let M_n be the number of words of length n (the length of a word is the number of letters) such that a letter b is never immediately followed by a letter c.

For example $M_2 = 8$:

$$aa, \ ab, \ ac, \ ba, \ bb, \ ca, \ cb, \ cc.$$

Question 1. Write a script which computes $M_1, M_2, \dots, M_{20}$ using a graph and its adjacency matrix. You can consider the following graph:



Answers.

Exercise 3. b-short words

Do it yourself. We say that a word w with letters a, b is b-*short* if there are never 4 consecutive b's in w. For instance,

$$w_1 = aabaaaaaabbbaaabaa$$

is b-short while

$$w_2 = aabaabbabbbbbbaabba$$

is not. Let S_n be the number of b-short words of length n (the length of a word is the number of letters).

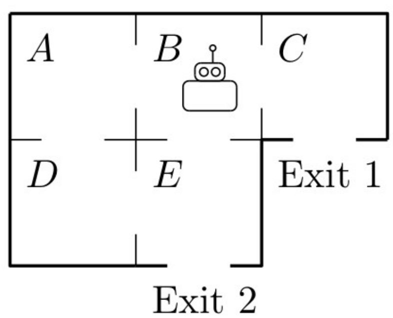
(To check your results: first values are `[2, 4, 8, 15, 29, ...`)

Answers.

Transition matrices

Exercise 4. Labyrinth.

We consider a random robot in the following labyrinth:



The robot is initially in room B (time $n = 0$). At each time step, it chooses uniformly at random one of the doors of the room in which it is located, and passes through that door.
If the robot hits Exit 1 (resp. 2) it stays at Exit 1 (resp. 2) forever.

Do it yourself.
Let $p(n)$ denote the vector of the probability distribution of the location of the robot at time n. More formally,

$$p(n) = \Big(p_x(n)\Big)_{x \in \{A,B,C,D,E,\text{Exit 1},\text{Exit 2}\}},$$

where $p_x(n)$ is the probability that the robot is at x at time n. Of course we have that

$$p(0) = (0, 1, 0, 0, 0, 0, 0).$$

1) Use a transition matrix M to compute approximate values of $p(5)$, $p(2000)$.

2) The robot eventually escapes the labyrinth, either through Exit 1 or Exit 2. Deduce from $p(2000)$ an approximation of the probability that the robot escapes the labyrinth through Exit 1.

Answers.

Answers.

Do it yourself. We still assume that the robot starts at B.
Denote by L_n the event "The robot is still in the labyrinth at time n" (*i.e.* it dit not find the exit yet). Use your matrix M to plot $n \mapsto \mathbb{P}(L_n)$. (Try $1 \leq n \leq 40$.)

Do it yourself. Use the method seen in class and the function `solve` from `SymPy` to compute the *exact* probability that starting from B the robot escapes the labyrinth through Exit 1.
Compare with your approximation obtained previously.

Recall that to solve a system with `solve` *you must write something like:*

```
var('x y')
solve([x-2*y,2*x-1+y],[x,y])
```

Answers.

Exercise 5. OK Corral

We consider the following probabilistic model. *(Its name refers to [this historical event (https://en.wikipedia.org/wiki/Gunfight_at_the_O.K._Corral)](https://en.wikipedia.org/wiki/Gunfight_at_the_O.K._Corral).)*

Initially there is a population of N gangsters, a of them belong to gang A and $b = N - a$ belong to gang B. At times $t = 0, 1, 2, \dots,$ a randomly and uniformly chosen gangster (among survivors) kills a member of the other gang. The process ends when one of the gangs is wiped out.

We want to find:

- The probability that the gang A wins (depending on a, b)
- The expected number of survivors of gang A at the end of the gunfight.

Do it yourself. Let $p(i, j)$ be the probability that Gang A wins the gunfight against gang B, starting from respectively i, j gangsters.

1. Find $p(i, j)$ when i or j equals 0.
2. For $i \geq 1$ and $j \geq 1$ find a recurrence relation between $p(i, j)$ and $p(i, j-1),\ p(i-1, j)$.
3. Write a function which computes $p(i, j)$ and draw a plot of $i \mapsto p(i, 8)$ for $1 \leq i \leq 16$.
   Is this consistent with intuition? In particular what is the value of $p(8, 8)$?

Answers.

Do it yourself. Let $e(i, j)$ be the expected numbers of survivors of Gang A after the gunfight starting from respectively i, j gangsters.

1. State a recursive formula for the $e(i, j)$'s.
2. Write a function which computes $e(i, j)$ and draw a plot of $i \mapsto e(i, 8)$ for $1 \leq i \leq 16$.

Answers.