Bachelor of Ecole Polytechnique
Computational Mathematics, year 2, semester 1
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Graphs and Probability 2: Solving some Probabilistic Models

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```
# execute this part to modify the css style
from IPython.core.display import HTML
def css_styling():
    styles = open("./style/custom2.css").read()
    return HTML(styles)
css_styling()
```

```
## loading python libraries

# necessary to display plots inline:
%matplotlib inline

# load the libraries
import matplotlib.pyplot as plt # 2D plotting library
import numpy as np              # package for scientific computing

from math import *              # package for mathematics (pi, arctan, sqrt, factorial
import sympy as sympy             # package for symbolic computation
from sympy import *
```

Model 1. OK Corral

We consider the following probabilistic model. *(Its name refers to this historical event (https://en.wikipedia.org/wiki/Gunfight_at_the_O.K._Corral).)*

Initially there is a population of N gangsters, a of them belong to gang A and $b = N - a$ belong to gang B. At times $t = 0, 1, 2, \dots,$ a randomly and uniformly chosen gangster (among survivors) kills a member of the other gang. The process ends when one of the gangs is wiped out.

We want to find:

- The probability that the gang A wins (depending on a, b)
- The expected number of survivors of gang A at the end of the gunfight.

Do it yourself. Let $p(i,j)$ be the probability that Gang A wins the gunfight against gang B, starting from respectively i, j gangsters.

1. Find $p(i,j)$ when i or j equals 0.
2. Prove that if $i \geq 1$ and $j \geq 1$ we have that
$$p(i,j) = \frac{i}{i+j}p(i,j-1) + \frac{j}{i+j}p(i-1,j). \qquad (\star)$$
3. Write a function which computes $p(i,j)$ and draw a plot of $i \mapsto p(i,b)$ for fixed $b=8$ and $1 \leq i \leq 2b$. Is this consistent with intuition?

Answers.

1.
2.

```
def GangA_wins(a,b):
    # return the probability that gang A wins starting from a,b individuals
```

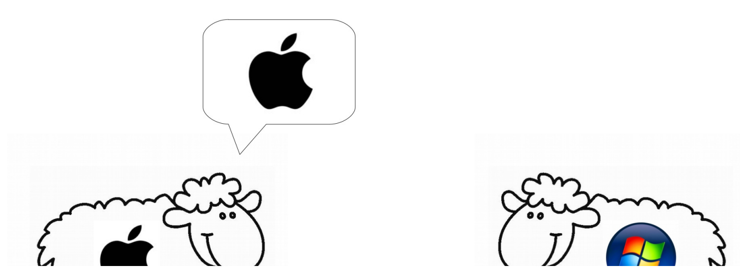
Do it yourself. Let $e(i,j)$ be the expected numbers of survivors of Gang A after the gunfight starting from respectively i, j gangsters.

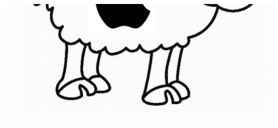
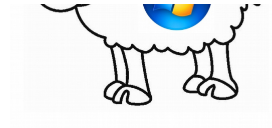
1. State a recursive formula for the $e(i,j)$.
2. Write a function which computes $e(i,j)$ and draw a plot of $i \mapsto e(i,b)$ for fixed $b=8$ and $1 \leq a \leq 2b$.

Answers.

```
def ExpectedSurvivorsA(a,b):
    # returns the expected number of survivals of gang A, starting from a,b individuals
```

Model 2. Opinion propagation in sheeps



Let $N > 1$ be fixed and consider a population of N sheeps, either pro-Mac or pro-Windows.

Initially, $0 \leq m \leq N$ sheeps are pro-Mac. At times $t = 0, 1, 2, \ldots,$ a randomly and uniformly chosen sheep bleats its opinion and instantly one sheep of the other camp switches its opinion. The process ends when unanimity is reached.

Do it yourself.

1. Let $M_k \in \{0, 1, \ldots, N\}$ be the number of pro-Mac sheeps at time k. Find the transition matrix $Q = (Q_{i,j})_{0 \leq i,j \leq N}$ associated to this Markov chain $(M_k)_k$.
2. Using Q, write a function `ProbaSheeps(N,m,i,t)` which computes the probability that starting from m, there are i pro-Mac sheeps at time t.
3. Draw a plot of i $\mapsto$`ProbaSheeps(N,m,i,t)` for $N = 100, m = 70, t = 30$.

Answers.

1.
2.

```
# Question 3
```

As 0 and N are absorbing states the processus is eventually absorbed at 0 or N: there is unanimity pro-Mac or pro-Windows among sheeps.

For fixed N let p_m denote the probability that the unanimity is achieved for Mac, starting from m pro-Mac and $N - m$ pro-Windows sheeps.

Remark. In order to compute p_m one could try to do the same strategy as we did for OK Corral and write a recursion for the p_m's. We would get:

$$p_m = \frac{N - (m+1)}{N} p_{m+1} + \frac{m-1}{N} p_{m-1}.$$

Unfortunately it is not easy to solve this recursion, therefore we will approximate p_m using the transition matrix.

Do it yourself.

1. Assume that the Markov chain (M_k) starts at $M_0 = m$. Explain why
$$p_m = \lim_{k \to +\infty} \mathbb{P}(M_k = N).$$
(Hint: If you want to write down a rigourous proof you may use Proposition 1.3 in Giovanni's Lecture Notes of MAA203 or [this link (Wikipedia) (https://en.wikipedia.org/wiki/Measure_(mathematics)#Continuity_from_below)](https://en.wikipedia.org/wiki/Measure_(mathematics)#Continuity_from_below).)
2. Use your function `ProbaSheeps(N,m,i,t)` to plot $m \mapsto p_m$ for $N = 20$. (We consider that $k = N^2$ is large enough for the previous approximation to hold.)

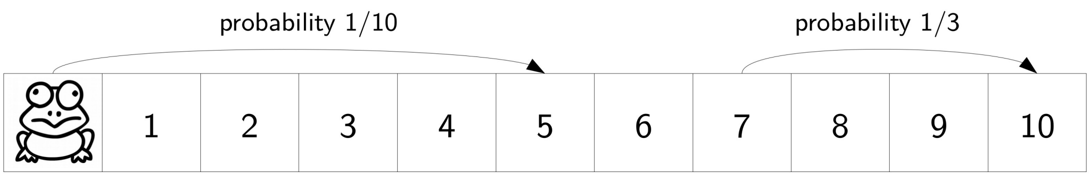
Answers. 1.

```
# Question 2
```

Model 3: The frog problem *(or how to become a trader?)*

This is a maths puzzle which used to be asked in job interviews (to be hired as a quantitative analyst or trader).

Here is the puzzle: a frog starts from the bank of a river located at $x = 0$. The other bank is located at $x = 10$. It first jumps uniformly at random in $\{1, 2, \dots, 10\}$ (if it jumps at 10 the frog has crossed the river and the process is over). Then, at each time step if it is located at y then it jumps at a uniform location in $\{y + 1, \dots, 10\}$.



The question is

What is the average number of steps before reaching the other bank of the river?

Do it yourself. Solve the Frog problem with python (or pen and paper, this is much harder but doable).
(Hint: Find a recursive relation for the sequence $e_0, e_1, \dots, e_{10}$ where N_i is the random variable

$$N_i = \text{ Number of steps before reaching } x = 10, \text{ starting from } i.$$

and where we put $e_i = \mathbb{E}[N_i]$.)

Answers.

```
def AverageFrog(i):
    # input: i in {0,1,...10}
    # returns the average number of steps starting from i
```

Bonus The birthday paradox

We consider the following problem. Consider a group of $n \geq 2$ people, we assume that their birthdays $X_1, \dots, X_n$ are uniformly distributed and independent in $\{1, 2, \dots, k\}$, with $k = 365$. The *birthday paradox* asks for the probability of the event

$$E_{n,k} = \{ \text{ there exist } i \neq j, 1 \leq i, j \leq n; X_i = X_j \}.$$

Obviously we have that $\mathbb{P}(E_{n,365}) = 1$ as soon as $n \geq 365$. The so-called *paradox* is that a high probability is reached for quite small values of n.

Do it yourself. Let $F_{n,k}$ be the complementary event of $E_{n,k}$.

1. Compute $\mathbb{P}(F_{1,k})$ and $\mathbb{P}(F_{2,k})$.
2. Compute
$$\mathbb{P}(F_{n,k} | F_{n-1,k}),$$
and deduce the formulas for $\mathbb{P}(F_{n,k})$, $\mathbb{P}(E_{n,k})$.

Answers. 1. 2.

Do it yourself. Write a function that takes n, k as inputs and returns $\mathbb{P}(E_{n,k})$.

```
def TwoIdenticalBirthdays(n,k):
    # returns the probability P(E_{n,k})
    #
    #
    #

# Test : (for n=8,k=365 this should return 0.0743...)
print('For n=8, two identical birthdays with probability '+str(TwoIdenticalBirthdays(8,
```

Do it yourself.

1. Plot $n \mapsto \mathbb{P}(E_{n,365})$ for $n = 2$ to $n = 100$.
2. Find the smallest n such that $\mathbb{P}(E_{n,365}) \geq 3/4$.

```
# Question 1


# Question 2
```

Answers. 2)

```

```