Exercise sheet - Week 14 (Probabilistic Toolbox)

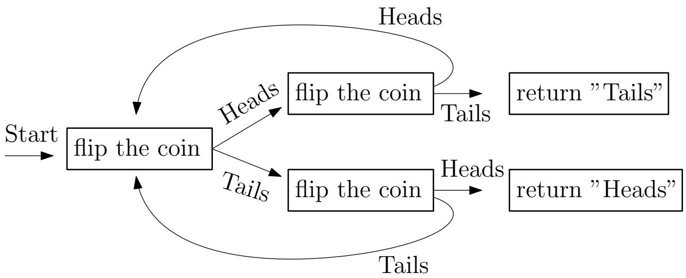
EXERCISE 1 -Reservoir Sampling

What is the output of the following algorithm ? (Assuming that successive calls of `rand()` return independent uniform random variables in (0,1).)

```
input: List x_1, ..., x_n
r = 0
for i = 1 to n:
   if rand()<1/i:
      r = x_i
return r
```

EXERCISE 2 -von Neumann's algorithm.

Consider an unfair coin that returns "tails" with probability $p \neq 1/2$. We want to use it to generate a fair heads or tails, John von Neumann[1] devised the following algorithm :



Let $\tau \in \mathbb{N} \cup \{+\infty\}$ be the random variable given by the number of throws required for the algorithm to stop, and $R \in \{\text{"Heads"}, \text{"Tails"}\}$ the output of the algorithm.

1. Give the values of τ and R if the first flips are $TTTTHHTTTHHT$?
2. For all $k \geq 1$, compute $\mathbb{P}(\tau = k)$. Deduce that the algorithm terminates almost surely : $\mathbb{P}(\tau < +\infty) = 1$.
3. Show that the algorithm returns "heads" or "tails" with equal probability, i.e. $\mathbb{P}(R = \text{"Heads"}) = 1/2$.
4. Show that $\mathbb{E}[\tau] = \frac{1}{p(1-p)}$.
5. (Bonus) How can the von Neumann algorithm be improved to reduce $\mathbb{E}[\tau]$?

EXERCISE 3 -Consecutive Heads

A fair coin is tossed infinitely many times. For $n \geq 1$, let M_n be the length of the longest sequence of consecutive "Heads" during the first n throws. For example :

n	1	2	3	4	5	6	7	8	9	...
$n-$ th toss	F	P	P	F	P	P	P	F	P	...
M_n	0	1	2	2	2	2	3	3	3	...

The aim of this exercise is to study the asymptotic behavior of the sequence (M_n).

1. Show that for all $k \leq n$ we have $\mathbb{P}(M_n \geq k) \leq (n-k+1)(1/2)^k$.
2. Deduce the asymptotic behaviour of (M_n) : for all $\varepsilon > 0$

$$\mathbb{P}\left(M_n \geq (1+\varepsilon)\log_2(n)\right) \overset{n \to +\infty}{\to} 0.$$

1. J.von Neumann. Various techniques used in connection with random digits. *Appl. Math Ser*, 12 (1951) p.36-38.