Exercise sheet - Week 15 (1st and 2d moment method)

EXERCISE 1 -Isolated vertices in $\mathcal{G}(n,p)$

Taken from : P.Brémaud (2017). *Discrete probability models and methods.* Springer.

Let $c > 0$ and consider a realization G_n of the Erdös-Rényi random graph $\mathcal{G}(n, p_n)$ where

$$p_n = c\frac{\log(n)}{n}.$$

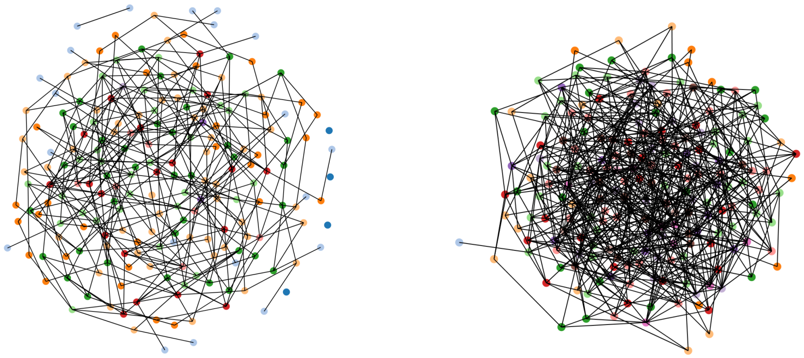
The aim of the exercise is to prove that

(i) If $c > 1$ then $\mathbb{P}(\text{no isolated vertices in } G_n) \overset{n\to+\infty}{\to} 1$.

(ii) If $c < 1$ then $\mathbb{P}(G_n \text{ has at least one isolated vertex}) \overset{n\to+\infty}{\to} 1$.

For a vertex $u \in G_n$ let $Z_u = \mathbf{1}_{u \text{ isolated}}$ and let $X_n = \sum_{u\in G_n} Z_u$ denote the number of isolated vertices in G_n.

1. Compute $\mathbb{E}[Z_u]$ and use the 1st moment method to prove (i).
2. Prove (ii) with the 2d moment method.



Two samples of $\mathcal{G}(n, p_n)$ with $n = 200$ and $p = 0.75\frac{\log(n)}{n}$ (left) and $p = 1.13\frac{\log(n)}{n}$ (right).
Source : `https://www.fitzner.nl/simulator/`.

EXERCISE 2 -Average characteristics in $\mathcal{G}(n,p)$

Let $p \in (0, 1)$ and consider a realization G_n of the Erdös-Rényi random graph $\mathcal{G}(n, p)$. Compute :

1. the average number of vertices of degree d.
2. the average number of cycles of length k (for some $k \le n$).

EXERCISE 3 -Balancing vectors

Taken from : S.Roch. *Modern Discrete Probability : An Essential Toolkit.* Cambridge Univ. Press (2024).

Let $v_1, \dots, v_d$ be arbitrary unit vectors in $\mathbb{R}^d$. Prove that there exist $x_1, \dots, x_d \in \{-1, +1\}$ such that

$$\| x_1v_1 + \cdots + x_dv_d \|_2 \le \sqrt{d},$$

where

$$\| y \|_2 = \sqrt{\langle y, y\rangle} = \sqrt{\sum y_i^2}.$$