Exercise sheet - Training session

EXERCISE 1 -Triangles in $\mathcal{G}(n,p)$.
For $n \geq 1$ and $p_n \in (0,1)$, consider a realization G_n of the Erdös-Rényi random graph $\mathcal{G}(n,p_n)$. Let $T_n \in \{0,1,\dots \binom{n}{3}\}$ be the random variable given by the number of triangles in G_n :

$$T_n = \sum_{1\leq i<j<k\leq n} \mathbf{1}_{\{i,j,k \text{ is a triangle in } G_n\}}.$$

1. Compute $\mathbb{E}[T_n]$.
2. Using the 1st moment method, prove that if $p_n = o(1/n)$ then $\mathbb{P}(G_n$ has no triangle$) \to 1$.
3. Using the 2d moment method, prove that if $1/n = o(p_n)$ then $\mathbb{P}(G_n$ has at least a triangle$) \to 1$.

EXERCISE 2 -Independent sets : an explicit bound.
In Lecture 15 we have seen an algorithm which takes as input a connected graph G and returns a random independent set $\mathcal{I}(G)$ in G. (Details of the algorithm are not relevant for this Exercise.) We proved that if G has n vertices and m edges then

$$\mathbb{E}\left[|\mathcal{I}(G)|\right] \geq \frac{n^2}{4m}.$$

By the "expectation argument" it implies that $p := \mathbb{P}(|\mathcal{I}(G)| \geq n^2/4m) > 0$.

1. Find a non-trivial lower bound for p.
   (Hint : write $\mathbb{E}\left[|\mathcal{I}(G)|\right] = \sum_{k<n^2/4m} k\mathbb{P}(|\mathcal{I}(G)| = k) + \sum_{k\geq n^2/4m} k\mathbb{P}(|\mathcal{I}(G)| = k)$.)
2. For $\varepsilon > 0$, how many times do we have to run the algorithm to guarantee an independent set greater than $n^2/4m$ with probability larger than $1-\varepsilon$?

EXERCISE 3 -Two balls in each bin : Back from Exercise Sheet 16.
One throws balls, one at a time, uniformly at random and independently into r bins. In Lecture 16 we proved that if $C = \min\{n \geq 1; \text{at least one ball in each bin at time } n\}$, then for all $\varepsilon > 0$ then

$$\mathbb{P}(C > r\log(r)(1+\varepsilon)) \to 0.$$

Let $\tilde{C} = \min\{n \geq 1; \text{at least two balls in each bin at time } n\}$, clearly $\tilde{C} > C$. However, prove that the same estimate as above still holds :

$$\mathbb{P}(\tilde{C} > r\log(r)(1+\varepsilon)) \to 0.$$