Events / Expectation

- Let $E_1, \ldots, E_n$ be events,
  - **(Union bound)** $\mathbb{P}(\cup_i E_i) \leq \sum \mathbb{P}(E_i)$
  - If E_i's are disjoint, $\mathbb{P}(\cup_i E_i) = \sum \mathbb{P}(E_i)$
- **(Conditioning)** $\mathbb{P}(E|F) = \frac{\mathbb{P}(E\cap F)}{\mathbb{P}(F)}$. More generally :

$$\mathbb{P}(E_1 \cap E_2 \cap \cdots \cap E_n) = \mathbb{P}(E_1) \times \mathbb{P}(E_2|E_1) \times \mathbb{P}(E_3|E_1 \cap E_2) \times \ldots \mathbb{P}(E_n|E_1 \cap E_2 \cap \cdots \cap E_{n-1}).$$

- **(Law of total probability)** Let $E_1, \ldots, E_n$ be disjoint such that $\Omega = E_1 \cup \cdots \cup E_n$. Then

$$\mathbb{P}(A) = \sum_i \mathbb{P}(A \cap E_i) = \sum_i \mathbb{P}(A|E_i)\mathbb{P}(E_i).$$

Expectation / Conditional expectation

- Let X be a random variable,

$$\mathbb{E}[X] = \sum_k k \times \mathbb{P}(X = k), \qquad \mathbb{E}[f(X)] = \sum_k f(k) \times \mathbb{P}(X = k),$$

  where the summation is over all values in the range of X.
- **(A useful formula)** Let $X \in \mathbb{Z}_{\geq 0}$,

$$\mathbb{E}[X] = \sum_{k \geq 0} \mathbb{P}(X > k).$$

- **(Linearity of expectations)** Let $X_1, X_2, \ldots, X_n, \ldots$ be a collection of random variables.
  - For every n, $\mathbb{E}[\sum_i^n X_i] = \sum_i^n \mathbb{E}[X_i]$.
  - If X_i's are non-negative, $\mathbb{E}[\sum_i^{+\infty} X_i] = \sum_i^{+\infty} \mathbb{E}[X_i]$.
- **(Conditional expectation)** $\mathbb{E}[X|Y = y] = \sum_k k\mathbb{P}(X = k \mid Y = y)$.
- **(Law of total expectations)** $\mathbb{E}[X] = \sum_y \mathbb{E}[X|Y = y]\mathbb{P}(Y = y)$.
- **(Conditional expectation : the abstract formula)** $\mathbb{E}[X|Y] = h(Y)$ where $h(y) := \mathbb{E}[X|Y = y]$.
- **(Tower property of conditional expectations)**

$$\mathbb{E}\left[\mathbb{E}[X|Y]\right] = \mathbb{E}[X].$$

Useful distributions

- **Uniform** distribution in $\{a, a+1, \ldots, b\}$, $\mathbb{P}(X = i) = 1/(b - a + 1)$ for each i.

$$\mathbb{E}[X] = \frac{a+b}{2}, \quad \mathrm{Var}(X) = \frac{(b-a+1)^2 - 1}{12}.$$

- **Bernoulli** distribution with mean $p \in [0, 1]$, $\mathbb{P}(X = 1) = p$

$$\mathbb{E}[X] = p, \quad \mathrm{Var}(X) = p(1 - p).$$

  (a.k.a. indicator random variables : $\mathbb{E}[\mathbf{1}_A] = \mathbb{P}(A)$.)
- **Binomial** distribution $\mathrm{Bin}(n, p)$ with parameters $n \geq 1, p \in [0, 1]$
  (= number of successes in n Bernoulli trials)
  $\mathrm{Bin}(n, p) \in \{0, 1, \ldots, n\}$ and

$$\mathbb{P}(\mathrm{Bin}(n, p) = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad \mathbb{E}[X] = np, \quad \mathrm{Var}(X) = np(1-p).$$

- by the way, a **useful estimate for binomial coefficients** :

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k.$$

- **Geometric** distribution $\mathcal{G}$ with parameter $p \in (0, 1)$
  (= first success in Bernoulli trials)
  $\mathcal{G} \in \{1, 2, \ldots\}$ and

$$\mathbb{P}(\mathcal{G} = k) = p(1-p)^{k-1}, \quad \mathbb{E}[X] = 1/p, \quad \mathrm{Var}(X) = (1-p)/p^2.$$